ON AN INVARIANT TRANSFORMATION OF EQUATIONS OF ONE-DIMENSIONAL UNSTEADY MOTION OF AN IDEAL COMPRESSIBLE FLUID

(OB ODNOM INVARIANTNOM PREOBRAZOVANII URAVNENII ODNOMERNYKH NEUSTANOVIVSHIKHSIA DVIZHENII IDEAL'NOI SZHIMAEMOI ZHIDKOSTI)

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L. A. MOVSESIAN
(Yerevan)

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The aim of this paper is to obtain transformations for the impulse and continuity equations of one-dimensional unsteady flows of an ideal gas, whose initial system is invariant with respect to them. This unables us to introduce a new flow plane $x_2 t_2$ in addition to initial space-time plane $x_1 t_1$ where the compressible fluid obeys a different equation of state which contains an arbitrary particle function (*). The connection established between the flows in the two space-time planes enables solutions to be found in one of the planes by simple calculation from a known solution in the other plane.

Presence of an arbitrary particle function in one of the planes allows us, in the case of adiabatic flows, to find a constant entropy flow, corresponding to isentropic flow only at the position of the particle and for the special equation of state.

1. Transformation of equations of motion. Let us consider Euler equations

$$\rho_1 \frac{\partial u_1}{\partial t_1} + \rho_1 u_1 \frac{\partial u_1}{\partial x_1} + \frac{\partial p_1}{\partial x_1} = 0, \qquad \frac{\partial \rho_1}{\partial t_1} + \frac{\partial}{\partial x_1}(\rho_1 u_1) = 0 \tag{1.1}$$

where p_1, ρ_1 and u_1 are pressure, density and velocity of a compressible fluid, respectively : x_1 and t_1 are space and time coordinates.

We easily see that relations

$$dx_2 = (1 + \chi\rho_1)\, dx_1 - \chi\rho_1 u_1\, dt_1, \qquad \frac{d\chi}{dt_1} = 0 \tag{1.2}$$

follow from the continuity equation and the condition that χ is maintained in the particle.

If $1 + \chi\rho_1 \neq 0$ in the given region of plane $x_1 t_1$, functions $x_2(x_1 t_1)$ and $t_2 = t_1$ are independent, because $D(x_2, t_2) / D(x_1, t_1) = 1 + \chi\rho_1 \neq 0$.

Let us put

$$p_2 = p_1, \quad \rho_2 = \frac{\rho_1}{1 + \chi\rho_1}, \quad u_2 = u_1 \tag{1.3}$$

If we are now changing over to the new independent variables x_2 and t_2 and are using (1. 3), we obtain an analogous system

*) Particle function is a function of a Lagrangian coordinate.

$$\rho_2 \frac{\partial u_2}{\partial t_2} + \rho_2 u_2 \frac{\partial u_2}{\partial x_2} + \frac{\partial p_2}{\partial x_2} = 0, \qquad \frac{\partial \rho_2}{\partial t_2} + \frac{\partial}{\partial x_2}(\rho_2 u_2) = 0 \tag{1.4}$$

Thus the motion of a fluid with parameters p_1, ρ_1 and u_1 in plane 1 (x, t_1) corresponds to the motion of some other fluid with parameters p_2, ρ_2 and u_2 within plane 2 $(x_2 t_2)$, and transition from plane 1 to plane 2 is effected by Formulas (1.2) and (1.3). The reverse transition is given by

$$dx_1 = (1 - \chi\rho_2)\, dx_2 + \chi\rho_2 u_2 dt_2 \tag{1.5}$$

$$p_1 = p_2, \qquad \rho_1 = \frac{\rho_2}{1 - \chi\rho_2}, \qquad u_1 = u_2 \tag{1.6}$$

It is clear that for one to one correspondence between planes 1 and 2, it is essential that the conditions $1 + \chi\rho_1 \neq 0$, $1 - \chi\rho_2 \neq 0$ hold. Direct and inverse transformation formulas show clearly that both planes obey the same rules, but they are not equivalent. This follows from the fact that ρ_1 can be arbitrarily large, while $\rho_2 < 1/\chi$.

2. Particle streamlines. It follows from the appropriate continuity equation that the particle functions $\psi_1(x_1, t_1)$ and $\psi_2(x_2, t_2)$ are connected with the flow parameters in both planes by the relations

$$\rho_1 = \frac{\partial \psi_1}{\partial x_1}, \qquad \rho_1 u_1 = -\frac{\partial \psi_1}{\partial t_1}, \qquad \rho_2 = \frac{\partial \psi_2}{\partial x_2}, \qquad \rho_2 u_2 = -\frac{\partial \psi_2}{\partial t_2}$$

These transformations lead one to conclude that if ψ_1 = const and ψ_2 = const are particle streamlines in the corresponding planes, then $\psi_1(x_1 t_1) = \psi_2(x_2 t_2)$, i.e. these transformations map ψ_1 = const in plane 1 into ψ_2 = const in plane 2. Furthermore at corresponding points in the two planes the angles between the tangents to the particle streamlines and the corresponding coordinate axes are equal, but the curvatures of the streamlines differ.

Observe that the streamline ψ_1 = const [ψ_2 = const], being a function of the Lagrangian coordinate $\xi_1(x_1, t_1)$ [$\xi_2(x_2, t_2)$] only, represents the law of motion of each particle and thus each given particle in plane 1 corresponds to one particle in plane 2. It follows, that, if χ is the particle function in plane 1, it retains the same significance in plane 2.

3. Interdependence of the solutions to problems in both planes. From a known solution of any given problem in plane 1, (1.2) can be used to determine x_2 as a function of x_1 and t_1

$$x_2 = \int_L (1 + \chi\rho_1)\, dx_1 - \chi\rho_1 u_1\, dt_1 + m_1 \tag{3.1}$$

where χ, ρ_1 and u_1 are known functions of x_1 and t_1 and the line integral should be taken along any contour L connecting the given points in plane 1.

Using relations (1.3) and (3.1) therefore it is easy to find a solution of the corresponding problem in plane 2 in the parametric form $p_2 = p_2(x_1, t_1)$, $\rho_2 = \rho_2(x_1, t_1)$, $u_2 = u_2(x_1, t_1)$, $x_2 = x_2(x_1, t_1)$, $t_2 = t_1$. From here, eliminating x_1 and t_1 we can write the solution of our problem in terms of x_2 and t_2.

Therefore, from the solution of any given problem in one plane, the solution of another problem in the other plane can be found by simple calculation.

From the condition $u_1 = u_2$ it follows, that the law of motion of a piston in both planes is the same. If we assume that $\rho_1 > 0$, $\rho_2 > 0$ and $\chi > 0$, then $\partial x_2/\partial x_1 > 0$.

Consequently, the positive and negative directions of corresponding coordinate axes coincide, and $x_1 \to \pm\infty$ when $x_2 \to \pm\infty$.

4. Equations of state. We shall assume adiabatic flow in both planes. We will take plane 1, in which an unsteady flow of fluid is completely known as the initial one. As the entropy s_1 is constant for each particle, we have $s_1 = s_1(\psi_1)$. Therefore the equation of state $p_2 = F\{\rho_2 / [1 - \chi(\psi_2)\rho_2], s_1(\psi_2)\}$ in plane 2 will correspond to the equation of state $p_1 = F(\rho_1, s_1)$ in plane 1. In particular, if we have isentropic flow in plane 1 $[p_1 = F(\rho_1)]$ the presence of an arbitrary particle function χ in the transformation formulas, limits the consideration of constancy of entropy in plane 2, to particles only.

In the case of linear relationship $p_1 = c_1^2 \rho_1$ (c_1 is the velocity of sound which is constant) the corresponding closing equation in plane 2 is expressed by Equation $p_2 = c_1^2\rho_2 / [1 - \chi(\psi_2)\rho_2]$. It should be noted that if $dp_1 / d\rho_1 > 0$, $d^2p_1 / d\rho_1^2 > 0$, then the corresponding relation is valid in plane 2. It is easy to see that the velocities of sound in two cases are connected by $c_2 = [1 + \chi(\psi_2)\rho_1]\, c_1$. If we know the solution of any given isentropic gas flow problem, exhibiting the relation $p_1 = F(\rho_1)$, we can establish a corresponding solution for another nonisentropic gas flow problem which is characterized by the relation $p_2 = F(\rho_2 / [1 - \chi(\psi_2)\rho_2])$, and for which more complex methods of solution are used.

If the connection $p_1 = F_1(\rho_1)$ is fixed, the corresponding relation $p_2 = F_2(\rho_2, \psi_2)$ will be completely defined.

If, instead of (1.3) we take Formulas

$$p_2 = \alpha p_1 + \beta, \qquad \rho_2 = \frac{\lambda\rho_1}{1 + \chi\rho_1} \qquad u_2 = \left(\frac{\alpha}{\lambda}\right)^{1/2} u_1$$

where α, β and λ are arbitrary constants, the form of system (1.4) will not change and therefore by choosing α, β and λ we can approximate the given relation $p_2 = F_2'(\rho_2, \psi_2)$ by using function $p_2 = F_2(\rho_2, \psi_2, \alpha, \beta, \lambda)$. Thus in order to approximate the Poisson adiabatic expression $p_2' = \theta(\psi_2)\rho_2'^{\gamma}$ (p_2', ρ_2' are dimensionless) in the neighborhood of some value ρ_{20}' we choose in plane 1 a function $p_1' = \alpha \exp(-1 / \rho_1')$ (p_1', ρ_1' are dimensionless). Then the relation between p_2' and ρ_2' in plane 2 is determined by the functions

$$p_2' = \theta(\psi_2)\alpha_1 \exp(-\beta_1 / \rho_2'), \qquad [\beta_1\chi(\psi_2) = \ln\theta(\psi_2)]$$

If we equate the values of the function and their first derivatives at point ρ_{20}' we arrive at the constants

$$\alpha_1 = \rho_{20}'^{\,\gamma} \exp\gamma \qquad \beta_1 = \gamma\rho_{20}'.$$

It should be observed that the approximation curve chosen in this manner will at any point ρ_{20}' have positive second derivative only for values of $\gamma > 2$.

It is evident that the Poisson adiabatic relation can be approximated not only close to the points, but within some interval of values of ρ_2'. For instance if $\gamma = 2.7$ and $\alpha_1 = \beta_1 = 1$, within the interval $0.2 \leq \rho_2' \leq 0.5$, both curves nearly coincide and $0.008 \leq p_2' \leq 0.14$. When $\gamma = 2$ a good approximation is obtained over the interval $0.3 \leq \rho_2 \leq 0.5$.

5. Flow with shock waves. If in plane 1 there is a line of discontinuity of the first kind of hydrodynamic quantities, a similar pattern of shock waves can be studied in plane 2.

From the impulse and continuity equations at the shock wave we have [1]

$$\rho_1' (V_1 - u_1')(u_1' - u_1'') = p_1' - p_1'', \qquad \rho_1' (V_1 - u_1') = \rho_1'' (V_1 - u_1'') \tag{5.1}$$

and from the expressions in (1.5), we obtain the conditions at its image in plane 2

$$\rho_2' (V_2 - u_2')(u_2' - u_2'') = p_2' - p_2'', \quad \rho_2' (V_2 - u_2') = \rho_2'' (V_2 - u_2'') \tag{5.2}$$

One prime denotes the region in front of the shock wave, double prime, the region behind it. V_1 and V_2 are shock wave velocities in planes 1 and 2 connected by equivalent expressions

$$V_2 = V_1 + \chi(\psi_2)\rho_1' (V_1 - u_1') = V_1 + \chi(\psi_2)\rho_1'' (V_1 - u_1'') \tag{5.3}$$

If the shock waves in both planes are propagated through stationary media $(u_1' = u_2' = 0,\ p_1' = p_1^0,\ \rho_1' = \rho_1^0)$, Expression (5.3) simplifies to

$$V_2 = [1 + \chi(\psi_2)\rho_1^0] V_1$$

Formulas (5.2) and (5.3) are correct only in the case when χ does not undergo a discontinuity at the shock wave. Otherwise only those regions should be compared, which lie between the piston and the shock wave and for the outside regions these transformations are invalid.

6. Flow of a nonhomogeneous compressible fluid behind a piston. Let us suppose that in plane 2 motion of the piston is given by a law $x_2 = -at_2^2/2$ $(a > 0)$, and the compressible fluid obeys the equation of state $p_2 = c^2\rho_2 / [1 - \chi(\psi_2)\rho_2]$. In plane 1 the piston will obey the same law $x_1 = -at_1^2/2$, whilst for the gas, $p_1 = c^2\rho_1$ will be valid.

If $u_1 = 0$, $p_1 = p_1^0$ and $\rho_1 = \rho_1^0$ on the characteristic $x_1 = ct_1$, the solution to the posed problem in plane 1 is [1]

$$u_1 = \sqrt{a^2t_1^2 + 2ax_1 + c^2} - (c + at_1), \qquad \rho_1 = \rho_1^0 \exp\frac{u_1}{c}, \qquad p_1 = c^2\rho_1$$

$$\chi = E\left(at_1 - \sqrt{a^2t_1^2 + 2ax_1 + c^2} - c\ln\frac{\sqrt{a^2t_1^2 + 2ax_1 + c^2} - c}{c}\right)$$

Here E is an arbitrary function of its own argument η, and is a general solution of Equation $d\chi/dt_1 = 0$.

In accordance with Formula (3.1) we have

$$x_2 = x_1 + \rho_1^0 \int_L E(\eta)\exp\frac{u_1}{c}\,dx_1 - E(\eta)\,u_1\exp\frac{u_1}{c}\,dt_1 \tag{6.1}$$

On the line $x_1 = ct_1$, we have $\eta = c\left(1 + \ln\dfrac{at_1}{c}\right)$

If we use one or another form of function $E(\eta) = f(t_1)$ on line $x_1 = ct_1$, we can determine from (6.1) the first characteristic $t_2 = t_2(x_2)$ in plane 2 (in general a curved one).

Because in the undisturbed region of plane 2 we have $p_2 = p_2^0$, and $u_2 = 0$, we have, on characteristic $t_2 = t_2(x_2)$

$$\rho_2 = \frac{\rho_1^0}{1 + \chi[t_2(x_2)]\rho_1^0}$$

and this is the initial density distribution as a function of x_2.

We have thus been able to determine the functions

$$x_2(x_1, t_1), \quad t_2 = t_1, \quad u_2(x_1, t_1), \quad p_2(x_1, t_1), \quad \rho_2(x_1, t_1),$$

which are indeed the solutions of the required problem in plane 2 in parametric form.

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